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FOCAL SETS OF 4-TRANSNORMAL TORI IN R^4

by

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DEDICATION

To my father and mother.

To my sisters and brothers.

To my husband.

To my supervised DR. kamal Al-Banaawi

To my teachers.

To my close friends.

God bless them.

ISLAAM AL-SARAYREH.

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In the Name of Allah, the Most Gracious, the Most Merciful. All praise and thanks are Allah's, the Lord of the Alamin, the Most Gracious, the Most Merciful, The Only Owner (and the Only Ruling Judge) of the Day of Recompense. You (Alone) we worship, and You (Alone) we ask for help, Guide us to the Straight Way. The Way of those on whom You have bestowed Your Grace, not (the way) of those who earned Your Anger, nor of those who went astray. Amen.

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ISLAAM AL-SARAYREH.

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Abstract

Focal Sets Of 4-Transnormal Tori In R^4

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Mu'tah University, 2012

Transnormal tori is a special type of transnormal manifolds in Euclidean spaces. The normal planes of such mathematical objects have special properties.

In this thesis we study the focal sets of 4- transnormal tori in R^4. We introduce examples of 4- transnormal tori in R^4, then we calculate their focal sets and for 4-transnormal embeddings of tori in R^4 which are not the usual product embeddings, different behavior is deduced.

الملخص

المجموعات البؤرية للعجلاتِ ذواتِ النقل العمودي من الرتبة الرابعة في R^4

إسلام الصرايرة

جامعة مؤتة، 2012

إن العجلاتِ ذواتِ النقل العموديِّ هي نوعٌ خاصٌ من مُتعدداتِ الشُّعبِ ذواتِ النقل العموديِّ في الفضاءاتِ الاقليديةِ. إن المستوياتِ العموديـــة لهـــذهِ المُجـــسماتِ الرياضيةِ لها خواصٌ معينه.

في هذهِ الاطروحه نَدْرسُ المجموعات البؤرية للعجلاتِ ذواتِ النقل العموديِّ مِنْ الرتبةِ الرابعةِ في R^4 . كذلكَ نُقدم أمثلةً علـــى هـــذهِ العجـــلاتِ ونحـــسب لهـــا المجموعات البؤرية. وبالنسبةِ للانغماسات للعجلات في R^4 غير الضربيه ســـيكون السلوك مختلفاً.

Chapter One

Introduction

1.1 Preface

This thesis is concerned with the focal sets of transnormal tori in R^4. Transnormal tori are examples of transnormal spherical partial tubes, an area of research that is a part of a general relatively new idea called transnormality. The concept of transnormality is due to S.A.Robertson [13,14,15].

In this thesis we take V as a smooth (C^∞) compact connected m-manifold without boundary, smoothly embedded in R^n for some $n > m$. We write $N(p)$ to mean the $(n-m)$-plane which is normal to V at p. The manifold V is transnormal in R^n whenever $q \in N(p)$ implies $N(q) = N(p)$ for all $p, q \in V$. The set $V \cap N(p)$ is called the *generating frame* of V at p.

In this thesis, we study transnormal tori in R^4. Mainly, our work is built on the work of S.A Robertson [13,14,15], S. Carter, and K. Al-Banawi [2,3,4]. The most recent work on transnormality has been done by Al-Banawi in the form of a study of focal points of transnormal tori in R^4 [5], and by W. Al-Awasa in the form of a study of transnormal manifold in Euclidean spaces [1].

The study of transnormal tori in R^4 is another step on the route of studying spherical partial tubes, which was introduced by S.Carter and A.West in [9]. The idea will be explained later, even though, it is about embedding M as a smooth connected m-manifold without boundary by a smooth function $f : M \to R^n$ into the Euclidean space R^n, $n = m + k$. Then we consider the normal bundle of M in which a smooth subbundle with some type fibre is embedded. If the end point map is restricted to that fibre, then we get a partial tube around f The manifold $f(M)$ is usually called the *base* of the partial tube. A partial tube is spherical if the fibre is a sphere.

1.2 Aims and Objectives

The main aim of this thesis is to calculate the focal sets of a product transnormal torus in R^4, and to calculate the focal sets of a non product transnormal torus in R^4, which is the essence of Chapter 4. The research is built on two main theorems regarding the calculation of focal points in differential geometry. Then the generating frames (polytopes) of such tori are shown. Other results and examples are included within the thesis.

1.3 Summary of Thesis

This thesis consists of four chapters including this chapter, the introduction.

Chapter **2** deals with the main concepts needed to understand the preliminaries of the work. Also it contains the differential geometry of calculating focal points. The main content of Chapter **2** as follows.

A brief description of manifolds is introduced. The concept of a critical point is a base within this thesis, so definitions and theorems concerning critical points are included. The geometry of surfaces is studied in the field of focal points.

The concept of the Euler characteristic as a main term in geometry is explained with stated theorems to support the content of this thesis. Later on it will be used to state a good result in Chapter 4.

Chapter **3** introduces the concept of a transnormal manifold as a base for the research in this thesis. Usually, a transnormal manifold is described by its generating frame or its generating polytope. Chapter **3** summarizes the literature of transnormal manifolds and explains many properties concerning generating frames of transnormal manifolds. Again the Euler characteristic plays a good role in the explanation of the idea of transnormality.

Chapter **4** contains the main results and contributions within this thesis. The ideas are mainly about the calculation of the focal sets of transnormal tori in R^4. Transnormal tori are studied in Chapter 4 with examples of different purposes. Both a product torus and a non product torus are included.

Chapter Two
Preliminaries

Throughout this thesis R^n will be the standard n-dimensional Euclidean space equipped with the standard topology. Also $(x_1, \mathbf{K}, x_n)$ will be a co-ordinate system .

2.1 Differentiable Manifolds

Let M be a topological space of dimension m, U be an open in M and $H: U \to R^m$, a homeomorphism . Then $\{U, H\}$ is called a *chart* on M.

Definition 2.1.1. Let M be a Hausdorff topological space of dimension m. Then M is an m-dimensional topological *manifold* if every point $p \in M$ is covered by a chart. An *atlas* is the set of all charts covering M .

One can consider the least number of charts covering M so that any addition of extra charts will be unnecessary. In this case the overlap of charts can be avoided and then a topological manifold will have a C^∞ smoothness, and hence will be called a differentiable m-manifold. For a more abstract definition see [7] .

If M is embedded in R^n, then at each point $p \in M$, there exists a *tangent* m - plane whose definition is a generalization of the definition of the directional derivative in advanced calculus. Also at $p \in M$, there exists a *normal* $(n - m)$-plane, which is the complement of the tangent plane in R^n .

2.2 Critical Points

Let M be a smooth (C^∞) m-manifold and $f : M \to R$ a smooth function. If $p \in M$, then $T_p R$ is the tangent plane of M at p and $T_{f(p)}R$ is the tangent plane of R at $f(p)$. The induced map of the tangent planes is $df_p : T_p M \to T_{f(p)} R$.

Definition 2.2.1 . A point $p \in M$ is a *critical point* of f if df_p is the zero map, or equivalently, $\frac{\partial f}{\partial x_i}(p) = 0$ for all $i = 1, \mathbf{K}, m$ where $(x_1, ..., x_m)$ is a co-ordinate system in a neighborhood of p in M . Other points of M are called *regular points* of f .

Definition 2.2.2 . A critical point is called *non-degenerate* iff the Hessian matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j}(p))$ is nonsingular. If p is a non-degenerate critical point of

f, then its *index* is the number of negative eigenvalues of the Hessian matrix at p. It should be noted that $0 \le index\ (p) \le m$.

Definition 2.2.3. [12] A smooth function $f : M \to R$ is a *Morse* function if all the critical points of f are non-degenerate.

Lemma 2.2.4. (Lemma of Morse) [12] Let f be a smooth real valued function on a smooth *m*-manifold and p a non-degenerate critical point of f. Then there is a local co-ordinate system $(x_1, \ldots, x_m)$ in a neighborhood U of p with $x_i(p) = 0$ for all $i = 1, \ldots, m$ and such that the identity

$$f(x_1, \ldots, x_m) = f(p) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_m^2$$

holds throughout U where i is the index of f at p.

It follows from the identity in Lemma 2.2.4 that the index of any local maximum point is m and the index of any local minimum point is 0. We finish this section with an example which explains the phrases included within it.

Example 2.2.5. Consider the function $f : R^2 \to R$ defined by

$$f(x, y) = -xy + x^2 + 4y^2 + 3x + 6y - 1.$$

Here

$$f_x = -y + 2x + 3,$$
$$f_y = -x + 8y + 6.$$

Now $f_x = f_y = 0$ at the point $p(-2, -1)$ which is a critical point.

Also

$$f_{xx} = 2, \quad f_{yy} = 8, \quad f_{xy} = f_{yx} = -1.$$

The Hessian matrix at the point p is

$$H = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}.$$

Now $\det(H) = 15$, so this matrix is nonsingular. Finally, p is a non-degenerate critical point.

2.3 Focal Points

Let M be a smooth m-manifold and $f : M \to R^n$ a smooth embedding of M into R^n. Let $V = f(M)$ and $p \in V$. Let $(x_1, \ldots, x_m)$ be a local co-ordinate system of M. Let T_pV be the tangent plane of V at $p \in V$. Then.

$$NB = \{(p, v) : p \in V, v \perp T_pV\}$$

is the *normal bundle* of V. It is an n -manifold differentiable embedded in R^{2n}. Let $h : NB \to R^n$ be the smooth map defined by

$$h(p,v) = p + v \quad .$$

The map h is called the *endpoint map*.

Definition 2.3.1. The point e in R^n is a *focal point* of (V, p) if $e = p + v$, where $(p, v) \in NB$ and the Jacobian matrix of h at (p, v) is a singular.

Definition 2.3.2. [10] The *first fundamental form* associated with the co-ordinate system is the symmetric matrix of real valued functions

$$\Gamma = (\langle \frac{\partial f}{\partial x_i} \ , \ \frac{\partial f}{\partial x_j} \rangle).$$

The *second fundamental form* for the normal direction v associated with the co-ordinate system is the symmetric matrix of real valued functions

$$\Pi_n = (\langle n, \frac{\partial^2 f}{\partial x_i \partial x_j} \rangle).$$

where v is a unit normal vector to V.

The eigenvalues of the matrix $\Gamma^{-1}\Pi_v$ are called the *principal curvatures* $k_1, \mathbf{K}, k_m$ of V in the normal direction v. The reciprocals $\frac{1}{k_1}, \mathbf{K}, \frac{1}{k_m}$ of these principal curvatures are called the *principal radii* of curvature.

Now consider the normal line L consisting of all $p + tv$ where v is a fixed unit vector perpendicular to T_pV and $t \in R$.

Lemma 2.3.3. [12] The point $p + tv$ is a focal point of (V, p) iff the matrix $\Gamma - t\Pi_v$ evaluated at p is singular.

Proof. Let $n_1, \ldots, n_{n-m}$ be $n - m$ unit vectors perpendicular to each other and perpendicular to T_pV. Let $(x_1, \ldots, x_m, y_1, \ldots, y_{n-m})$ be the coordinates of the manifold $NB \subset V \times R^n$. Then a point in NB is of the form

$$< f(x_1, \mathbf{K}, x_m), \sum_{k=1}^{n-m} y_k v_k (x_1, \mathbf{K}, x_{m)} > .$$

The function $h : NB \to R^n$ maps such a point to

$$f(x_1 \mathbf{K}, x_m) + \sum_{k=1}^{n-m} y_k n_k (x_1, \mathbf{K}, x_m) \in R^n.$$

The first partial derivatives of h are

$$\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{k=1}^{n-m} y_k \frac{\partial n}{\partial x_i}, \qquad i = 1, \mathbf{K}, m,$$

$$\frac{\partial h}{\partial y_j} = n_j, \qquad j = 1, \mathbf{K}, n - m.$$

Taking the inner products of these n vectors with the vectors $\frac{\partial f}{\partial x_1}, \mathbf{K}, \frac{\partial f}{\partial x_m}$, $n_1, \mathbf{K}, n_{n-n}$, the result is an $n \times n$ matrix whose rank is equal to the rank of the Jacobian matrix of h at the corresponding point. This $n \times n$ matrix has the form

$$\begin{bmatrix} <\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}> + <\sum_{k=1}^{n-m} y_k \frac{\partial v_k}{\partial x_i}, \frac{\partial f}{\partial x_j}> & (<\sum_{k=1}^{n-m} y_k \frac{\partial v_k}{\partial x_i}, v_j>) \\ 0 & I_{n-m} \end{bmatrix}$$

and so its rank is equal to the rank of the upper left hand block. But

$$0 = \frac{\partial <v_k, \frac{\partial f}{\partial x_j}>}{\partial x_i} =< \frac{\partial v_k}{\partial x_i}, \frac{\partial f}{\partial x_j}> + <v_k, \frac{\partial^2 f}{\partial x_i \partial x_j}>,$$

and so the upper left hand block is just the matrix

$$<\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}> - <\sum_{k=1}^{n-m} y_k v_k, \frac{\partial^2 f}{\partial x_i \partial x_j}>.$$

Or $\Gamma - t\Pi_v$ where $tn = \sum_{k=1}^{n-m} y_k n_k$. Thus, the Jacobian of h at (p, tn) is singular iff the matrix $\Gamma - t\Pi_v$ evaluated at p is singular. ◊

Theorem 2.3.4. [12] the focal points of (V, p) along L are precisely the points $p + \frac{1}{k_i} v$ where $i = 1, \mathbf{K}, m, k_i \neq 0$.

Proof. By Lemma 2.3.3, e is a focal point of (V, p) iff $e = p + tv$ and the matrix $\Gamma - t\Pi_v$ evaluated at p is singular iff the matrix $I_m - t\Gamma^{-1}\Pi_v$ evaluated at p is singular iff the matrix $\Gamma^{-1}\Pi_v$ evaluated at p has $\frac{1}{t}$ as an eigenvalue. Thus, the values of t are just $\frac{1}{k_1}, \mathbf{K}, \frac{1}{k_m}$. ◊

In the next few examples, we explain how to use the above ideas to calculate the focal points of surfaces (two dimensional manifolds).

Example 2.3.5. Let $f(q, f) = (\cos q \sin f, \sin q \sin f, \cos f)$.
Now

$$f_q = (-\sin q \sin f, \cos q \sin f, 0),$$
$$f_f = (\cos q \cos f, \sin q \cos f, -\sin f).$$

Thus,

$$< f_q, f_q >= \sin^2 f,\ < f_q, f_f >=< f_f, f_q >= 0,\ < f_f, f_f >= 1.$$

The first fundamental matrix is $\Gamma = \begin{bmatrix} \sin^2 f & 0 \\ 0 & 1 \end{bmatrix}$.

Also

$$f_{qq} = (-\cos q \sin f, -\sin q \sin f, 0),$$
$$f_{fq} = (-\sin q \cos f, \cos q \cos f, 0),$$
$$f_{ff} = (-\cos q \sin f, -\sin q \sin f, -\cos f).$$

A unit normal to f is $n = \dfrac{f_q \times f_f}{\left\| f_q \times f_f \right\|} = (-\cos q \sin f, -\sin q \sin f, -\cos f)$.

Thus, the second fundamental matrix is

$$\Pi_v = \begin{bmatrix} \sin^2 f & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus,

$$\Gamma^{-1}\Pi_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalues of $\Gamma^{-1}\Pi_v$ are 1, 1. Thus at $f(q, f)$ the corresponding foci are repeated, each focal point is

$$f(q, f) + v(q, f) = (0,0,0).$$

Example 2.3.6. Let $f(q, f) = ((5 + 3\cos q)\cos f, (5 + 3\cos q)\sin f, 3\sin q)$.

Now

$$f_q = (-3\sin q \cos f, -3\sin q \sin f, 3\cos q),$$
$$f_f = (-(5 + 3\cos q)\sin f, (5 + 3\cos q)\cos f, 0).$$

Thus,

$$< f_q, f_q >= 9,\ < f_q, f_f >=< f_f, f_q >= (5 + 3\cos q)^2,\ < f_f, f_f >= 0.$$

The first fundamental matrix is $\Gamma = \begin{bmatrix} 9 & 0 \\ 0 & (5 + 3\cos q)^2 \end{bmatrix}$.

Also

$$f_{qq} = (-3\cos q \cos f, -3\cos q \sin f, -3\sin q),$$

$$f_{fq} = (-(5+3\cos q)\cos q, -(5+3\cos q)\sin f, 0),$$

$$f_{ff} = (3\cos q \sin f, -3\sin q \cos f, 0).$$

A unit normal to f is $n = \dfrac{f_q \times f_f}{\left\| f_q \times f_f \right\|} = (-\cos q \cos f, -\cos q \sin f, -\sin q)$.

Thus, the second fundamental matrix is

$$\Pi_v = \begin{bmatrix} 3 & 0 \\ 0 & (5+3\cos q)\cos q \end{bmatrix}.$$

So

$$\Gamma^{-1}\Pi_v = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{\cos q}{(5+3\cos q} \end{bmatrix}.$$

The eigenvalues of $\Gamma^{-1}\Pi_v$ are $\frac{1}{3}, \frac{\cos q}{(5+\cos q)}$. Thus, at $f(q, f)$, the corresponding foci are

$$f(q,f) + 3v(q,f) = (5\cos f, 5\sin f, 0)$$

and

$$f(q,f) + \frac{(5+3\cos q)}{\cos q} v(q,f) = (0,0,-5\tan q).$$

Example 2.3.7. Let $f(q,f) = (\cos q, \sin q, f)$.

Now

$$f_q = (-\sin f, \cos f, 0),$$
$$f_f = (0,0,1).$$

Thus,

$$< f_q, f_q >= 1, \ < f_q, f_f >=< f_f, f_q >= 0, \ < f_f, f_f >= 1.$$

The first fundamental matrix is $\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Also

$$f_{qq} = (-\cos q, -\sin q, 0),$$

$$f_{fq} = (0,0,0),$$

$$f_{ff} = (0,0,0).$$

A unit normal to f is $n = \dfrac{f_q \times f_f}{\| f_q \times f_f \|} = (\cos q, \sin q, 0)$.

Thus, the second fundamental matrix is

$$\Pi_v = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

So

$$\Gamma^{-1}\Pi_v = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of $\Gamma^{-1}\Pi_v$ are $-1, 0$. Thus, the first focal point is

$$f(q,f) - \frac{1}{1} v(q,f) = (0,0,v).$$

and the other focal point is infinity.

Now consider for a fixed point $p \in R^n$ the function $\Lambda_p : M \to R^+ \cup \{0\}$ where $\Lambda_p(x) = \| f(x) - p \|^2$. The function Λ_p is called the *distance function.*

Theorem 2.3.8. [12] The point $q \in V$ is a degenerate critical point of Λ_p iff p is focal point of (V, q).

Proof. The first partial derivatives of Λ_p are given by

$$\frac{\partial \Lambda_p}{\partial x_i} = 2 < \frac{\partial f}{\partial x_i}, f - p >, \quad i = 1,\dots, m.$$

The second partial derivatives of Λ_p are given by

$$\frac{\partial^2 \Lambda_p}{\partial x_i \partial x_j} = 2(< \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} > + < \frac{\partial^2 f}{\partial x_i \partial x_j}, f - p >), \quad i, j = 1,\dots, m.$$

Thus, q is a degenerate critical point of Λ_p iff $q - p \perp T_P V$ and the Hessian matrix of Λ_p at q, $(\dfrac{\partial^2 \Lambda_p}{\partial x_i \partial x_j}(q))$, is singular iff $p = q + tv$ for $t \in R$ and the matrix $\Gamma - t\Pi_v$ evaluated at q is singular iff p is a focal point of (V, q). ◊

Theorem 2.3.9. [12] The index of Λ_p at a non-degenerate critical point $q \in V$ is equal to the number of focal points of (V, q) which lie on the segment from *q* to *p*, each focal point being counted with its multiplicity.

Proof. From the proof of Theorem 2.3.8., the index of Λ_p at q is the number of negative eigenvalues of the matrix $\Gamma - s\Pi_v$ evaluated at q

where $p = q + sv$ for some real value s. Choose the coordinates such that Γ evaluated at q is the identity matrix I_m. Then the index of Λ_p at q is the number of negative solutions of the equation

$$\det((1 - l)I_m - s\Pi_v) = 0$$

where the equation is in the variable l, Π_v is evaluated at q and *det* is the determinant. The solutions of the above equation are of the form

$$\frac{1-l}{s} = k_i \qquad ,i = 1,.\mathbf{K},m.$$

So $l < 0$ iff $\frac{1}{k_i} < s$ iff l corresponds to a focal point that lies between q and p. ◊

Now we recalculate the focal points of the surfaces in the last examples using the distance function.

Example 2.3.10. Let $f(q,f) = (\cos q \sin f, \sin q \sin f, \cos f)$. Take $p = (a, b, c)$. Then

$$\begin{aligned}\Lambda_p &= \|f(q,f) - p\|^2 \\ &= (\cos q \sin f - a)^2 + (\sin q \sin f - b)^2 + (\cos f - c)^2 .\end{aligned}$$

Now

$$\frac{\partial \Lambda_p}{\partial q} = 2\sin f(a \sin q - b\cos q),$$

$$\frac{\partial \Lambda_p}{\partial f} = -2\cos f(a\cos q + b\sin q) + 2c\sin f,$$

$$\frac{\partial^2 \Lambda_p}{\partial q^2} = 2\sin f(a\cos q + b\sin q),$$

$$\frac{\partial^2 \Lambda_p}{\partial f^2} = 2\sin f(a\cos q + b\sin q) + 2c\cos f,$$

$$\frac{\partial \Lambda_p}{\partial q\ \partial f} = 2\cos f(a\sin q - b\cos q).$$

So $f(q,f)$ is a degenerate critical point of Λ_p iff

$$\sin f(a\sin q - b\cos q) = 0 \qquad (2.1.a)$$

$$\cos f(a\cos q + b\sin q) - c\sin f = 0 \qquad (2.1.b)$$

$$\begin{aligned}&4\sin^2 f(a\cos q + b\sin q)^2 + 4c\sin f\cos f(a\cos q + b\sin q) \\ &- 4\cos^2 f(a\sin q - b\cos q)^2 = 0\end{aligned} \qquad (2.1.c)$$

Solving the system (2.1), we have $a = b = c = 0$. Thus, the focal point is $(0,0,0)$.

Example 2.3.11. Let $f(q,f)=((5+3\cos q)\cos f,(5+3\cos q)\sin f,3\sin q)$. Take $p=(a,b,c)$. Then

$$\Lambda_{\mathrm{p}}=\|f(q,f)-p\|^2$$
$$=((5+3\cos q)\cos f-a)^2+((5+3\cos q)\sin f-b)^2+(3\sin q-c)^2.$$

Now

$$\frac{\partial\Lambda_p}{\partial q}=6[(a\cos f+b\sin f-5)\sin q-c\cos q],$$
$$\frac{\partial\Lambda_p}{\partial f}=2(5+3\cos q)[a\sin f-b\cos f],$$
$$\frac{\partial^2\Lambda_p}{\partial q^2}=6[(a\cos f+b\sin f-5)\cos q+c\sin q],$$
$$\frac{\partial^2\Lambda_p}{\partial f^2}=2(5+3\cos q)[a\cos f+b\sin f],$$
$$\frac{\partial^2\Lambda_p}{\partial q\,\partial f}=6\sin q\,[b\cos f-a\sin f].$$

Now $f(q,f)$ is a degenerate critical point of Λ_p iff

$$(a\cos f+b\sin f-5)\sin q-c\cos q=0 \tag{2.2.a}$$

$$(5+3\cos q)[a\sin f-b\cos f]=0 \tag{2.2.b}$$

$$\begin{aligned}&12(a\cos f+b\sin f)[(5\cos q(a\cos f+b\sin f-5)+5c\sin q\\&+(3\cos^2 q(a\cos f+b\sin f-5)+3c\sin q\cos q)]\\&-36\sin^2 q(b\cos f-a\sin f)^2=0\end{aligned} \tag{2.2.c}$$

By (2.2.b)

$$a\sin f-b\cos f=0 \tag{2.2.d}$$

and so Equation (2.2.c) becomes

$$(a\cos f+b\sin f)((a\cos f+b\sin f-5)\cos q+c\sin q)=0 \tag{2.2.e}$$

If

$$a\cos f+b\sin f=0 \tag{2.2.f}$$

then solve Equations (2.2.d), (2.2.f) together to get $a=b=0$, and then by Equation (2.2.a) we get $c=-5\tan q$. Thus, the focal point is $(0,0,-5\tan q)$.

If $$(a\cos f + b\sin f - 5)\cos q + c\sin q = 0 \qquad (2.2.g)$$

then solve Equations (2.2.d), (2.2.g) together to get $c = 0$,and

$$a\cos f + b\sin f - 5 = 0 \qquad (2.2.h)$$

Now solve Equations (2.2.d), (2.2.h) together to get $a = 5\cos f, b = 5\sin f$. Then the focal point is on the circle $(5\cos f, 5\sin f, 0)$.

Example 2.3.12. Let $f(q,f) = (\cos q, \sin q, f)$. Take $p = (a,b,c)$. Then

$$\begin{aligned}\Lambda_p &= \|f(q,f) - p\|^2 \\ &= (\cos q - a)^2 + (\sin q - b)^2 + (f - c)^2\end{aligned}$$

Now

$$\frac{\partial \Lambda_p}{\partial q} = 2a\sin q - 2b\cos q,$$
$$\frac{\partial \Lambda_p}{\partial f} = 2(f - c),$$
$$\frac{\partial^2 \Lambda_p}{\partial q^2} = 2a\cos q + 2b\sin q,$$
$$\frac{\partial^2 \Lambda_p}{\partial f^2} = 2,$$
$$\frac{\partial^2 \Lambda_p}{\partial q\,\partial f} = 0.$$

Now $f(q,f)$ is a degenerate critical point of Λ_p iff

$$a\sin q - b\cos q = 0 \qquad (2.3.a)$$

$$f - c = 0 \qquad (2.3.b)$$

$$a\cos q + b\sin q = 0 \qquad (2.3.c)$$

Solving the system (2.3), we have $a=b=0$, $c=f$. Thus, the focal point is $(0,0,f)$.

2.4 The Euler Characteristic

The Euler characteristic of a manifold is an alternating sum over the cells of the manifold. To have a simple look , for a surface M the Euler characteristic is defined by

$$c(M) = a - b + c$$

where a is the number of *vertices* of M , b is the number of *edges* of M ,and c is the number of *faces* of M .

Definition 2.4.1. A closed m -manifold is a smooth(C^{∞}) compact connected m -manifold without boundary .

Let C_i denote the number of critical points of f of index i.

Theorem 2.4.2. [12] Let M be a closed m -manifold . Let $f : M \to R$ be a Morse function . Then

$$c(M) = \sum_{i=0}^{m} (-1)^i C_i .$$

Remark 2.4.3. [6] This is a list of the Euler characteristics of the sphere S^n, the connected sum of n tori , nT^2, the connected sum of n projective planes, nP^2.

$$c(S^{2n}) = 2, \quad n \geq 0,$$
$$c(S^{2n+1}) = 0, \quad n \geq 0,$$
$$c(nT^2) = 2 - 2n, \quad n > 0,$$
$$c(nP^2) = 2 - n, \quad n > 0.$$

As the Klein bottle K^2 is the connected sum of two projective planes [6], $c(K^2) = 0$.

Definition 2.4.4. Let X be a topological space . A *covering space* of X is a pair consisting of a space $\overline{X}$ called a cover and a continuous map $P : \overline{X} \to X$ called a *covering map* such that for each $p \in X$ there is a path wise connected open neighborhood U of p such that each component of $P^{-1}(U)$ is mapped homeomorphically to U by P. The space X is usually called the base space of covering space.

Theorem 2.4.5. [11] Let $(\overline{X}, P)$ be a covering space of a space X . Then the sets $P^{-1}(p)$ for all $p \in X$ have the same cardinal number.

Geometrically, the cardinality of the set $P^{-1}(p)$ is the number of *sheets (folds)* of the covering space.

Definition 2.4.6. Let $(\overline{X}, P)$ be a covering space of a space X. Then the order r of $(\overline{X}, P)$ is the number of *sheets (folds)* of $(\overline{X}, P)$.

Theorem 2.4.7. [11] Let $(\overline{X}, P)$ be a covering space of a space X. Then $\overline{X}$ is an n-manifold iff X is an n-manifold.

Theorem 2.4.8. [11] Let $(\overline{X}, P)$ be an r-sheeted covering function of the space X. Then $c(\overline{X}) = r\ c(X)$.

Chapter Three
Euclidean Transnormality

The idea of transnormality is a generalization of the concept of an m-hypersurface of constant width in R^{m+1}, it is due to S.A.Robertson [13,14,15] and contributions have been made by B.Wegner [16,17,18,19], S.Carter and K.AL-Banawi [2,3,4,5].

3.1 Transnormality

The concept of constant width can be studied as follows. Let M be a smooth compact connected m-manifold without boundary that is smoothly embedded in R^{m+1}. A chord of M is *normal* if it is normal to M at one of its endpoints and *binormal* if it is normal to M at both endpoints. The manifold M is of constant width if and only if every *normal* chord of M is *binormal* to M. Each point of the endpoints is called the *opposite* of the other. Curves of constant width were introduced by Euler in 1778 [10]. The differential geometry of curves and surfaces of constant width was studied by Al-Banawi [2].

Let M be a smooth connected $m-$manifold without boundary and let $f: M \rightarrow R^n$ be a smooth embedding of M into R^n. Let $V = f(M)$. For each point $p \in V$ there exists a unique tangent plane T_pV tangent to V at p with dimension m and a unique normal plane N_pV normal to V at p with dimension $n - m$. Thus, there are maps T and N with
$T(p) = T_pV$ and $N(p) = N_pV$.

Definition 3.1.1. [13] The m-manifold V is *transnormal* in R^n iff

$$\forall p, q \in V, \text{ if } q \in N(p) \text{ then } N(q) = N(p).$$

Now define the relation ~ on V by saying that $p \sim q$ iff $q \in N(p)$, then ~ is an equivalence relation. The factor space $V/\sim$ can be identified with the space of normal planes of V, say $W = N(V)$.

Robertson [13] showed that the map $N: V \rightarrow W$, assigning to each the normal $(n-m)$-plane $N(p)$, is a covering map. A detailed proof of the fact that N is covering map was given by Wegner [16]. By Theorem 2.4.7, W is a manifold of dimension m. Wegner [18] showed that if V is a transnormal embedding of a curve in R^n, then the order of N is finite. It was just in 1981 when Wegner [19] showed that for any transnormal embedding V in R^n the order of N is always finite. If V is transnormal in R^n and the order of N is r, then V is called an r-transnormal manifold.

Definition 3.1.2.[13] Let V be a transnormal manifold in R^n. The *generating frame* of V at p is

$$f(p) = V \cap N(p).$$

If V is r-transnormal, then $|f(p)| = r$ where $|\quad|$ is the cardinality.

It is clear that any two generating frames are isometric. That is, if $f(p_1)$ and $f(p_2)$ are generating frames , then there exists $\mathbf{l} : f(p_1) \to f(p_2)$ which preserves distance.

Any normal plane of a compact r-transnormal manifold V cuts V transversally at exactly r points. Also there are planes of the same dimension as these normal planes that don't intersect V at all. The mod 2 intersection number of each plane with V is therefore equal to zero. Since any two planes of the same dimension can be moved onto one another, it follows that the mod 2 intersection number of any normal plane with V is also 0. Thus, if V is a compact r-transnormal manifold, then r is even[14].

Definition 3.1.3. [14] The *generating polytope* of a transnormal manifold is the convex hull of the generating frame of V.

3.2 Distance function on a transnormal manifold

Let M be an r-transnormal m-manifold in R^n. Let $x = (x_1, \ldots, x_m)$ be local co-ordinates of M. Assume that the inclusion map from M to R^n is defined by f. Take $V = f(M)$. For $p \in V$, let $\Lambda_p : M \to R^+ \cup \{0\}$ be the distance function defined by $\Lambda_p(x) = \|f(x) - p\|^2$. Then Λ_p has a critical point at q iff $q - p$ is in $N(p)$.

For,

$$\Lambda_p(x) = (f(x) - p) \cdot (f(x) - p), \quad 1 \le i \le m.$$

So

$$\frac{\partial \Lambda_p(x)}{\partial x_i} = 2\frac{\partial f}{\partial x_i} \cdot (f(x) - p), \quad 1 \le i \le m \ .$$

Now $\dfrac{\partial \Lambda_p(q)}{\partial x_i} = 0$ iff $q - p \perp \dfrac{\partial f}{\partial x_i}$, $1 \le i \le m$ iff $q - p \in N(q)$.

Thus, the set of critical points of Λ_p is the generating frame $f(p)$.

Theorem 3.2.1. [16] A transnormal manifold does not intersect its focal points.

By Theorem 3.2.1 and Theorem 2.3.8, the generating frame $f(p)$ is the set of non-degenerate critical points of Λ_p, i.e , Λ_p is Morse function.

The Euler characteristic of compact transnormal manifold

If M is compact and C_i denotes the number of critical points of Λ_p of index i, then the Euler characteristic of M is given by $c(M) = \sum_{i=0}^{m} (-1)^i C_i$ (Theorem.2.4.2).

Since $N : M \to W$ is an r-fold covering, then $c(M) = rc(W)$. Also $C_0 > 0$ so that $-r < c(M) \leq r$, and hence $-1 < c(W) \leq 1$. So $c(W) = 0$ or $c(W) = 1$. Thus, for compact r-transnormal manifold either $c(M) = 0$ or $c(M) = r$.

Thus, using Remark 2.4.3, one of the conclusions is that:
"Up to homeomorphism the sphere S^2, the torus T^2 and the Klein bottle K^2 are the only compact transnormal surfaces that can exist".

In R^4, $r = 2$ iff M is homeomorphic to S^2 [13]. So the chance of finding other transnormal embeddings of compact surfaces in R^4 with $r > 2$ is reduced to T^2 and K^2.

Theorem 3.2.2. [5] There does not exist a transnormal embedding of the Klein bottle K^2 in R^4.

In[5], using [8], AL-Banawi showed that 4-transnormal compact surfaces in R^4 are tori.

Theorem 3.2.3. [1] Let M be an r-transnormal manifold in R^4 that is homeomorphic to torus . If W the space of normal planes to M , is orientable, then $r = 4$ and the generating polytope is a rectangle, possibly a square.

Chapter Four
Focal Sets of 4-Transnormal Tori in R^4

4.1 Introduction to partial tubes

The general definition of a *partial tubs* was introduced in [9].

For an abstract definition of a partial tube about a smooth connected m-manifold M in R^n, let $f: M \to R^n$ be a smooth embedding of M into R^n, where $n = m + k$. For $p \in M$, let $T_p M$ be the tangent plane of M at p. Consider the normal bundle of M

$$NB = \{(p,v) : p \in M, v \perp T_p M\}$$

and the smooth endpoint map $h : NB \to R^n$ defined by $h(p,v) = p + v$. Let Ω be the set of focal points of $f(M)$ (singular points of h). Take $P \subset NB$ as a subbundle with type fiber S such that S is a smooth submanifold of R^k. Assume that $P \cap \Omega = \phi$. Then $h \mid P$ is a smooth embedding called a *partial tube* around f. The manifold $f(M)$ is called the *base* of the partial tube. A partial tube is *spherical* if the type fibre S is a sphere.

4.2 Transnormal partial tubes

We start with M as a closed m-manifold. Then we embedding of M into R^n by the smooth function f. Now assume that $f(M)$ is r-transnormal. The aim of the next theorem is to give a guarantee that there exists $x > 0$ such that the above partial tube is an embedding. Also if $p \in f(M)$, then the normal plane of $f(M)$ at p is $N(p)$, intersects the partial tube at points based at the points of the generating frame $f(p)$. Let $NB^x f(M) = \{(p,v) \in NB : p \in f(M) \; and \; \|v\| \le x\}$.

Theorem 4.2.1. [2] Let $f: M \to R^n$ be a smooth r-transnormal embedding of the closed m-manifold M into the Euclidean space R^n. Then there exists $x > 0$ sufficiently small such that

1) The map $h \mid NB^x f(M)$ is an embedding, and

2) For all $p \in f(M)$, for all $(q,v) \in h \mid NB^x f(M)$,

$$h(q,v) \in N(p) \text{ iff } q \in N(p) \cap f(M).$$

Theorem 4.2.2. [2] Let $f: M \to R^{m+d+k}$ be a smooth r-transnormal embedding of the closed m-manifold M into the Euclidean space R^{m+d+k}, $m \geq 1$, $d \geq 1$, $k \geq 2$. Assume that the dimension of the generating polytope of $f(M)$ is d. Then

1) There exists a $2r$-transnormal embedding of a $(k-1)$-sphere bundle over $f(M)$ in R^{m+d+k} with image a partial tube and $f(M)$ is its base.

2) The generating polytope of the partial tube is induced by two copies of the generating polytope of its base each parallel to the generating polytope of the base with a distance of the radius of the $(k-1)$-sphere from it.

Now we state and prove a main theorem regarding the order of a transnormal torus in R^n, $n > 4$.

Recall that C_m is the number of critical points of index m.

Theorem 4.2.3. If F is an r-transnormal embedding of a torus in R^n, $n > 4$, such that $C_2 = C_0$, then $r = 4k$ where $k \in \mathrm{N}$.

Proof. Since $c(T^2) = 0$, we have

$$C_0 - C_1 + C_2 = 0.$$

But $C_0 = C_2$. Then $C_1 = 2C_0$. Let $k = C_0$. Then the order of transnormality is

$$r = C_0 + C_1 + C_2 = k + 2k + k = 4k. \Diamond$$

Example 4.2.4. Let $F(q,f)=(\cos q,\sin q,\cos f,\sin f)$ where $q\in[0,2p]$, $f\in[0,p]$. Take $F(0,0)=(1,0,1,0)$. Then

$$\Lambda_{(1,0,1,0)}=\|F(q,f)-(1,0,1,0)\|^2$$
$$=4-2\cos q-2\cos f.$$

Now

$$\frac{\partial\ \Lambda_{(1,0,1,0)}}{\partial q}=2\sin q\ ,\qquad \frac{\partial\ \Lambda_{(1,01,0)}}{\partial\ f}=2\sin f\ ,$$

$$\frac{\partial^2\Lambda_{(1,01,0)}}{\partial\ q^2}=2\cos q\ ,\qquad \frac{\partial^2\Lambda_{(1,01,0)}}{\partial\ f^2}=2\cos f\ ,\qquad \frac{\partial^2\ \Lambda_{(1,0,1,0)}}{\partial q\,\partial f}=0..$$

Critical points of $\Lambda_{(1,0,1,0)}$ are at : $(0,0),(0,p),(p,0),(p,p)$.

The Hessian matrix of $\Lambda_{(1,0,1,0)}$ at $(0,0)$ is $\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$.

The Hessian matrix of $\Lambda_{(1,0,1,0)}$ at $(0,p)$ is $\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}$.

The Hessian matrix of $\Lambda_{(1,0,1,0)}$ at $(p,0)$ is $\begin{bmatrix}-2 & 0\\ 0 & 2\end{bmatrix}$.

The Hessian matrix of $\Lambda_{(1,0,1,0)}$ at (p,p) is $\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}$.

Thus, $\Lambda_{(1,0,1,0)}$ has 1 critical point of index 0, namely at $(1,0,1,0)$.

Also $\Lambda_{(1,0,1,0)\,)}$ has 2 critical points of index 1, namely at $(1,0,-1,0)$ and $(-1,0,1,0)$. Finally, $\Lambda_{(1,0,1,0)}$ has 1 critical point of index 2, namely at $(-1,0,-1,0)$. Now the generating polytope of F is as in Figure 1.

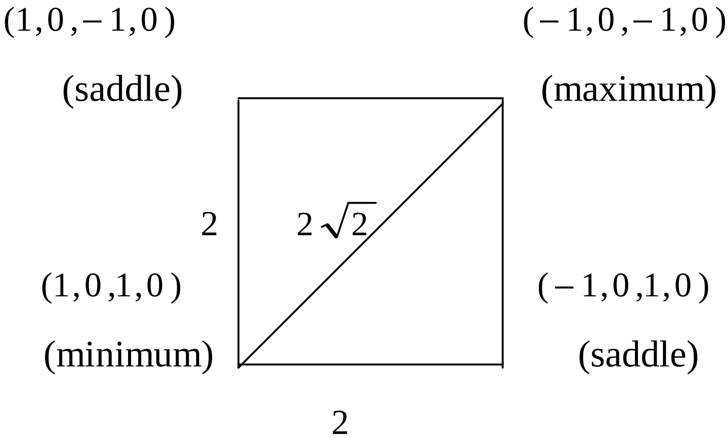


Figure 1 : The generating polytope of

$F(q,f)=(\cos q,\sin q,\cos f,\sin f)$.

Example4.2.5. Let $F(q,f)=(4\cos q, 4\sin q, 3\cos f, 3\sin f)$ where $q\in[0,2p]$, $f\in[0,p]$.

Take $F(0,0)=(4,0,3,0)$. Then

$$\Lambda_{(4,0,3,0)} = \left\|F(q,f)-(4,0,3,0)\right\|^2$$
$$= 50-32\cos q-18\cos f.$$

Now

$$\frac{\partial\Lambda_{(4,0,3,0)}}{\partial q}=32\sin q\,,\qquad \frac{\partial\Lambda_{(4,0,3,0)}}{\partial f}=18\sin f\,,$$

$$\frac{\partial^2\Lambda_{(4,0,3,0)}}{\partial q^2}=32\cos q\,,\qquad \frac{\partial^2\Lambda_{(4,0,3,0)}}{\partial f^2}=18\cos f\,,\qquad \frac{\partial^2\Lambda_{(4,0,3,0)}}{\partial q\,\partial f}=0\,.$$

Critical points of $\Lambda_{(4,0,3,0)}$ are at : $(0,0),(0,p),(p,0),(p,p)$.

The Hessian matrix of $\Lambda_{(4,0,3,0)}$ at $(0,0)$ is $\begin{bmatrix}32 & 0\\ 0 & 18\end{bmatrix}$.

The Hessian matrix of $\Lambda_{(4,0,3,0)}$ at $(0,p)$ is $\begin{bmatrix}32 & 0\\ 0 & -18\end{bmatrix}$.

The Hessian matrix of $\Lambda_{(4,0,3,0)}$ at $(p,0)$ is $\begin{bmatrix}-32 & 0\\ 0 & 18\end{bmatrix}$.

The Hessian matrix of $\Lambda_{(4,0,3,0)}$ at (p,p) is $\begin{bmatrix}-32 & 0\\ 0 & -18\end{bmatrix}$.

Thus , $\Lambda_{(4,0,3,0)}$ has 1 critical point of index 0, namely at $(4,0,3,0)$

Also $\Lambda_{(4,0,3,0)}$ has 2 critical points of index 1, namely at $(4,0,-3,0)$ and $(-4,0,3,0)$.

Finally $\Lambda_{(4,0,3,0)}$ has 1 critical point of index 2, namely at $(-4,0,-3,0)$ Now the generating polytope of F is as in Figuer 2.

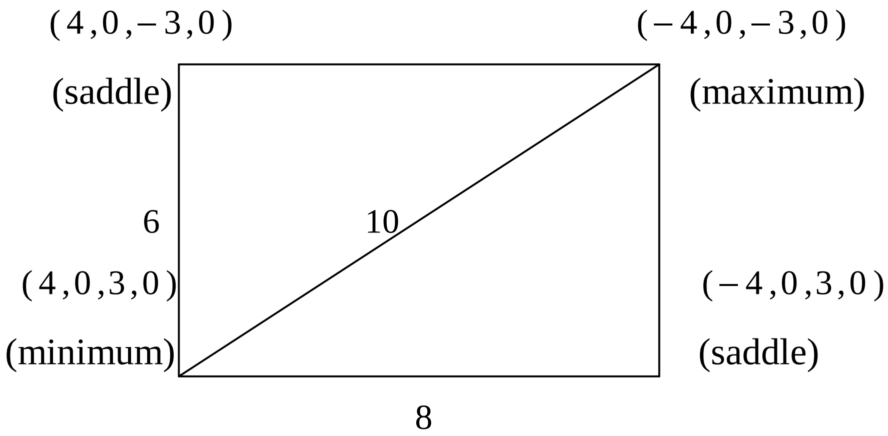


Figure 2 : The generating polytope of

$F(q,f)=(4\cos q,4\sin q,3\cos f,3\sin f)$.

4.3 General examples

Now we introduce two particular examples of transnormal tori in R^4. These examples were discussed by Al-Banawi in [5].

Example 4.3.1. Consider the 2 -transnormal embedding f of S^1 in R^{1+1+2} defined by

$$f(q)=(\cos q,\sin q,0,0)\,,\quad q\in[0,2p\,].$$

An orthonormal set of normals to f is given by

$$v_1(q)=-f(q),\quad v_2(q)=(0,0,1,0),\quad v_3(q)=(0,0,0,1).$$

Now a torus in R^4 can be defined as the product of f by itself. The torus is 4 -transnormal as it is the partial tube

$$h(q,f)=f(q)+v_2(q)\cos f+v_3(q)\sin f$$

which is

$$h(q,f) = (\cos q, \sin q, \cos f, \sin f), \qquad q \in [0,2p], \quad f \in [0,2p].$$

Now let

$$\Lambda(q,f) = (x - \cos q)^2 + (y - \sin q)^2 + (z - \cos f)^2 + (w - \sin f)^2 .$$

Then

$$\frac{1}{2}\frac{\partial \Lambda}{\partial q} = x\sin q - y\cos q ,$$

$$\frac{1}{2}\frac{\partial \Lambda}{\partial f} = z\sin f - y\cos f .$$

Also

$$\frac{1}{2}\frac{\partial^2 \Lambda}{\partial q^2} = x\cos q + y\sin q ,$$

$$\frac{1}{2}\frac{\partial^2 \Lambda}{\partial f^2} = z\cos f + w\sin f ,$$

$$\frac{1}{2}\frac{\partial^2 \Lambda}{\partial q\,\partial f} = 0 .$$

Now (x, y, z, w) is a focal point of h with base (q_1, f_1) iff

$$x\sin q_1 - y\cos q_1 = 0 \tag{4.1}$$

$$z\sin f_1 - w\cos f_1 = 0 \tag{4.2}$$

$$(x\cos q_1 + y\sin q_1)(z\cos f_1 + w\sin f_1) = 0. \tag{4.3}$$

If $x\cos q_1 + y\sin q_1 = 0$ and $z\cos f_1 + w\sin f_1 = a$, then Equation (4.1) implies that $x = y = 0$ and Equation (4.2) implies that $z = a\cos f_1, w = a\sin f_1$.

If $x\cos q_1 + y\sin q_1 = a$ and $z\cos f_1 + w\sin f_1 = 0$, then Equation (4.2) implies that $z = w = 0$ and Equation (4.1) implies that $x = a\cos q_1$, $y = a\sin q_1$.

So the lines of focal points of h at an arbitrary base (q_1, f_1) are

$$x = y = 0, \quad z = a\cos f_1, \quad w = a\sin f_1,$$

and $$z = w = 0, \quad x = a\cos q_1, \quad y = a\sin q_1,$$

which are perpendicular.

Example 4.3.2. Now consider the 2 -transnormal embedding f of S^1 in R^{1+1+2} defined by

$$f(q) = (\sin q, \cos q, \frac{1}{3}\sin 3q, \frac{1}{3}\cos 3q), \qquad q \in [0, 2p].$$

The embedding f of $S^1 \times S^1$ is 4-transnormal in R^4 which is not the product embedding.

An orthonormal set of normals to f is given by

$$v_1(q) = \frac{-3}{\sqrt{10}} f(q),$$

$$v_2(q) = \frac{1}{\sqrt{2}}(-\cos q, \sin q, \cos 3q, -\sin 3q),$$

$$v_3(q) = \frac{3}{\sqrt{10}}(-\frac{1}{3}\sin q, -\frac{1}{3}\cos q, \sin 3q, \cos 3q).$$

Let $x > 0$ be the radius of the 1 -sphere bundle over f. Then the 4-transnormal torus is

$$h(q, f) = (h_1, h_2, h_3, h_4),$$

where

$$h_1 = \sin q - \frac{x}{\sqrt{2}}\cos q \sin f - \frac{x}{\sqrt{10}}\sin q \sin f,$$

$$h_2 = \cos q + \frac{x}{\sqrt{2}}\sin q \cos f - \frac{x}{\sqrt{10}}\cos q \sin f,$$

$$h_3 = \frac{1}{3}\sin 3q + \frac{x}{\sqrt{2}}\cos 3q \cos f + \frac{3x}{\sqrt{10}}\sin 3q \cos f,$$

$$h_4 = \frac{1}{3}\cos 3q - \frac{x}{\sqrt{2}}\sin 3q \cos f + \frac{3x}{\sqrt{10}}\cos 3q \sin f.$$

Now let

$$\Lambda(q, f) = (x - h_1)^2 + (y - h_2)^2 + (z - h_3)^2 + (w - h_4)^2.$$

Then at $q=0,\ f=0,$ we have

$$\frac{1}{2}\frac{\partial \Lambda}{\partial q}=\frac{3x}{\sqrt{2}}(w-\frac{1}{3}y)-(x+z)\,,$$

$$\frac{1}{2}\frac{\partial \Lambda}{\partial f}=\frac{x}{\sqrt{10}}(y-3w)\,.$$

Also

$$\frac{1}{2}\frac{\partial^2 \Lambda}{\partial q^2}=y-\frac{x}{\sqrt{2}}x+\frac{9x}{\sqrt{2}}z+3w\,,$$

$$\frac{1}{2}\frac{\partial^2 \Lambda}{\partial f^2}=\frac{x}{\sqrt{2}}(z-x)\,,$$

$$\frac{1}{2}\frac{\partial^2 \Lambda}{\partial q\,\partial f}=\frac{x}{\sqrt{10}}(x-9z)\,.$$

Now (x, y, z, w) is a focal point of h with base $(0,0)$ iff

$$\frac{3x}{\sqrt{2}}(w-\frac{1}{3}y)-(x+z)=0 \qquad (4.4)$$

$$x(y-3w)=0 \qquad (4.5)$$

$$\frac{x}{\sqrt{2}}(y-\frac{x}{\sqrt{2}}x+\frac{9x}{\sqrt{2}}z+3w)(z-x)=\frac{9x^2}{10}(\frac{1}{3}x-3z)^2 \qquad (4.6)$$

Now Equation (4.4) and Equation (4.5) imply that $w=\frac{1}{3}y$ and $x=-z$.

Substitute for x and w into Equation (4.6) to get

$$10x^2z^2+2\sqrt{2}x\,y\,z=10x^2z^2 \qquad (4.7)$$

Thus ,

$$2\sqrt{2}x\,y\,z=0 \qquad (4.8)$$

Thus, Equation (4.8) implies that either $z=0$ or $y=0$. So the lines of focal points of h with base $(0,0)$ are

$$x=z=0\,,\qquad w=\frac{1}{3}y\,,$$

and

$$y=w=0\,,\qquad x=-z\,,$$

which are perpendicular.

Now at $q = 0, f = \frac{p}{2},$ we have

$$\frac{1}{2}\frac{\partial \Lambda}{\partial q} = \frac{x}{\sqrt{10}}(x-9z)-(x+z),$$

$$\frac{1}{2}\frac{\partial \Lambda}{\partial f} = \frac{x}{\sqrt{2}}(z-x),$$

$$\frac{1}{2}\frac{\partial^2 \Lambda}{\partial q^2} = \frac{x}{\sqrt{10}}(27w-y)+(3w+y),$$

$$\frac{1}{2}\frac{\partial^2 \Lambda}{\partial f^2} = \frac{x}{\sqrt{10}}(3w-y),$$

$$\frac{1}{2}\frac{\partial^2 \Lambda}{\partial q\,\partial f} = \frac{x}{\sqrt{2}}(y-3w)\ .$$

Now (x, y, z, w) is a focal point of h with base $(0,\frac{p}{2})$ iff

$$\frac{x}{\sqrt{10}}(x-9z)-(x+z)=0 \qquad (4.9)$$

$$x(z-x)=0 \qquad (4.10)$$

$$\left(\frac{3x}{\sqrt{10}}(y+3w)+\frac{x^2}{10}(81w-3y)\right)(w-\frac{1}{3}y)=\frac{9x^2}{2}(w-\frac{1}{3}y)^2 \qquad (4.11)$$

Now Equation (4.9) and Equation (4.10) imply that $x = z = 0$.

By Equation (4.11), either $w = \frac{1}{3}y$ or

$$\frac{3x}{\sqrt{10}}(y+3w)+\frac{x^2}{10}(81w-3y)-\frac{9x^2}{2}(w-\frac{1}{3}y)=0 \qquad (4.12)$$

which is equivalent to

$$\left(\frac{9x^2}{2}-\frac{9x}{\sqrt{10}}-\frac{81x^2}{10}\right)w=\left(\frac{3x^2}{2}+\frac{3x}{\sqrt{10}}-\frac{3x^2}{10}\right)y \qquad (4.13)$$

Now the two lines of the focal point of h with base $(0,\frac{p}{2})$ are

perpendicular iff

$$\left(\frac{9x^2}{2}-\frac{9x}{\sqrt{10}}-\frac{81x^2}{10}\right)=-\left(\frac{x^2}{2}+\frac{x}{\sqrt{10}}-\frac{x^2}{10}\right) \qquad (4.14)$$

iff

$$\frac{10x^2}{2}-\frac{82x^2}{10}-\frac{8x}{\sqrt{10}}=0 \qquad (4.15)$$

By Equation (4.15) we have

$$x=-\frac{\sqrt{10}}{4}$$

which is false as $x>0$. Thus, the two lines of the focal points of h with base $(0,\frac{p}{2})$ are not perpendicular.

As seen from the above examples, in the case of an embedding of a 4-transnormal product torus in R^4, the lines of focal points at every point of the embedding are perpendicular. Such a property is missed when the embedding of a 4-transnormal torus in R^4 is not the usual product one, it is explained by a point on the embedding where the lines of focal points were not perpendicular.

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